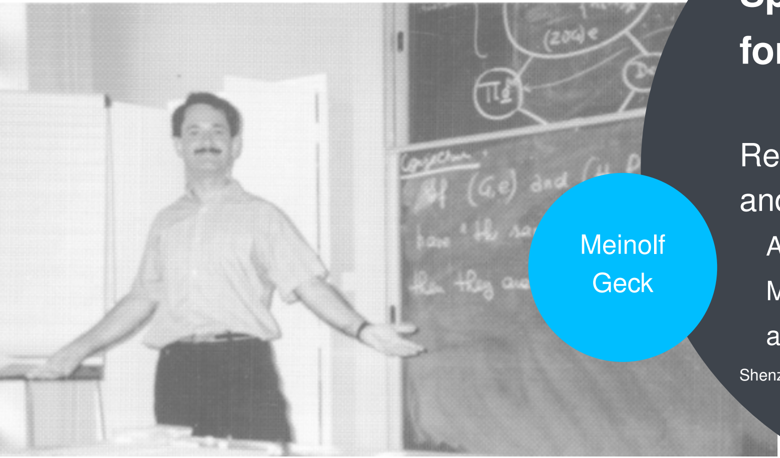
Universität Stuttgart

Special Chevalley bases for simple Lie algebras



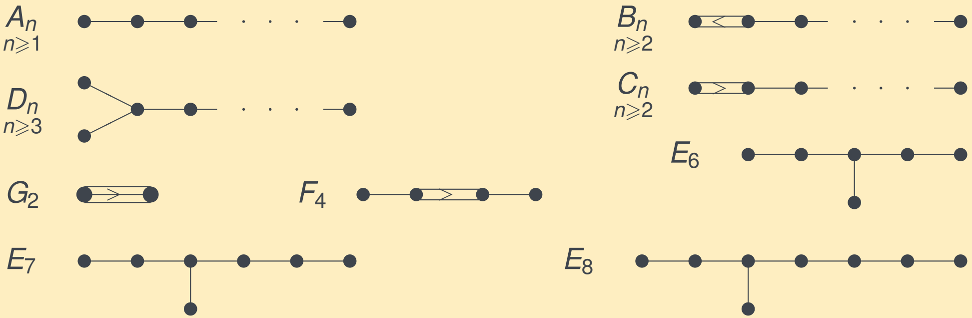
Meinolf Geck

Representations of finite groups and ramifications

A conference in honour of
Michel Broué’s 80th birthday
and his fundamental contributions

Shenzhen Center for Mathematics, May 24 - 29, 2026

Cartan–Killing (~1888/1894): The finite-dimensional simple Lie algebras over $\mathbb{C}$ are classified by "Dynkin diagrams".



Infinite families:

$$A_n \leftrightarrow \mathfrak{sl}_{n+1}(\mathbb{C}), \quad B_n \leftrightarrow \mathfrak{so}_{2n+1}(\mathbb{C}), \quad C_n \leftrightarrow \mathfrak{sp}_{2n}(\mathbb{C}), \quad D_n \leftrightarrow \mathfrak{so}_{2n}(\mathbb{C}).$$

Exceptional algebras:

$$\dim \mathfrak{g}_2 = 14, \quad \dim \mathfrak{f}_4 = 52, \quad \dim \mathfrak{e}_6 = 78, \quad \dim \mathfrak{e}_7 = 133, \quad \dim \mathfrak{e}_8 = 248.$$

By the way (as an anecdote) . . .

According to A. J. Coleman, The Mathematical Intelligencer **11** (1989), 29–38:
"***The greatest mathematical paper of all time***" is

W. KILLING, *Die Zusammensetzung der stetigen, endlichen Transformationsgruppen II*, Mathematische Annalen **33** (1888), 1–48.

MathSciNet Review by J. Dieudonné:
"Many mathematicians will disagree, [...]. One may however observe that:

(1) nobody had tackled the problem before Killing;
(2) he solved it by methods he invented;
(3) nobody collaborated with him for that solution;
(4) the result became a most important milestone in modern mathematics.

I think it is not so easy to find papers exhibiting all those features."

Bases for simple Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with $\dim \mathfrak{g} < \infty$.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra and $\Phi \subseteq \mathfrak{h}^*$ the corresponding root system.

Then $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ with $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi$.

Let $\{\alpha_i \mid i \in I\}$ be a system of simple roots (where $I \leftrightarrow$ Dynkin diagram).

Chevalley (1955). There is a basis $\{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ of $\mathfrak{g}$ such that

(1) $\mathfrak{h} = \langle h_i \mid i \in I\rangle_\mathbb{C}$ and $e_\alpha \in \mathfrak{g}_\alpha$ for all $\alpha \in \Phi$.

(2) $\langle e_{\alpha_i}, e_{-\alpha_i}, h_i\rangle_\mathbb{C} \cong \mathfrak{sl}_2(\mathbb{C})$ for $i \in I$.

(3) $[e_\alpha, e_\beta] = \pm(q_{\alpha,\beta} + 1)e_{\alpha+\beta}$ whenever $\alpha, \beta, \alpha+\beta \in \Phi$, where

$$q_{\alpha,\beta} := \max\{k \geqslant 0 \mid \beta - k\alpha \in \Phi\} \in \{0, 1, 2\}.$$

(4) All other Lie brackets determined by properties of the root system Φ.

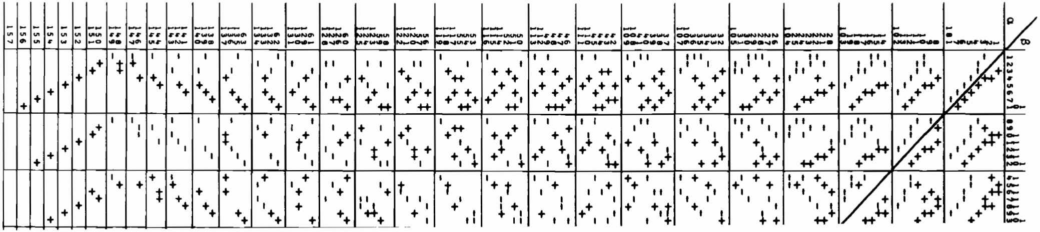
Subject of this talk: The $\{e_\alpha\}$ are only determined up to signs.

$\exists$ Canonical choice of e_α's ? Formula for "structure signs" $\pm$ in (3) ?

What would this be good for ?

Example: MIZUNO (1980) determined representatives of the unipotent classes of the Chevalley groups of type E_8 (a very impressive piece of work).

He needed to fix a Chevalley basis of the underlying Lie algebra $\mathfrak{g}$ and then his results are expressed with respect to that basis.



What has been known so far?

- TITS (1996): "*Sur les constantes de structures ...*"

  $\rightsquigarrow$ general theory for "structure signs", also yields proof of existence of $\mathfrak{g}$.

- FRENKEL–KAC (1980): "*Basic representations of affine Lie algebras ...*"

  $\rightsquigarrow$ for simply laced types, much simpler description and existence theorem.

- RINGEL (1990): "*Hall polynomials for ...*"

  $\rightsquigarrow$ new approach, generalises Frenkel–Kac to non simply-laced types.

- LUSZTIG (1990): General theory of "canonical bases".
  - Every finite-dimensional highest weight module for $\mathfrak{g}$ has a canonical basis.
  - Regard $\mathfrak{g}$ as a $\mathfrak{g}$-module via adjoint representation $\rightsquigarrow$ canonical basis of $\mathfrak{g}$.
  - "*The canonical basis of ...*" (2017) $\rightsquigarrow$ Identification of canonical basis of $\mathfrak{g}$.

- G. (2016): "*On the construction of ...*" $\rightsquigarrow$ elementary description of Lusztig's basis of $\mathfrak{g}$; see also *lecture notes* at `arxiv:2404.11472`.

Special (= Lusztig's) Chevalley bases

Fix $\langle e_{\alpha_i}, e_{-\alpha_i}, h_i\rangle_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$ for all $i \in I$ "Chevalley generators".

Then this extends to a collection $\{e_\alpha \mid \alpha \in \Phi\}$ such that, for each $i \in I$, we have

$$[e_{\alpha_i}, e_\alpha] = c_i(q_{\alpha_i,\alpha} + 1)e_{\alpha+\alpha_i} \quad \text{for all } \alpha \in \Phi \text{ such that } \alpha + \alpha_i \in \Phi,$$

$$[e_{-\alpha_i}, e_\alpha] = c_i(q_{-\alpha_i,\alpha} + 1)e_{\alpha-\alpha_i} \quad \text{for all } \alpha \in \Phi \text{ such that } \alpha - \alpha_i \in \Phi,$$

where the sign $c_i = \pm 1$ only depends on i.

The collection $\{e_\alpha \mid \alpha \in \Phi\}$ is unique up to a global sign.

What about the "global sign" and the signs c_i ?

Assume that $i \neq j$ are joined by an edge in the Dynkin diagram.

Then $\alpha_i + \alpha_j \in \Phi$ and $q_{\alpha_i,\alpha_j} = q_{\alpha_j,\alpha_i} = 0$. So

$$[e_{\alpha_i}, e_{\alpha_j}] = c_i e_{\alpha_i+\alpha_j} \quad \text{and} \quad [e_{\alpha_j}, e_{\alpha_i}] = c_j e_{\alpha_j+\alpha_i} \qquad \Rightarrow \qquad c_i = -c_j.$$

Hence, $i \mapsto c_i$ alternating along the nodes of the Dynkin diagram.

Dynkin diagram is a tree $\Rightarrow$ only two choices for the function $i \mapsto c_i$.

Each of these two choices determines a unique collection $\{e_\alpha \mid \alpha \in \Phi\}$ as above.

Dynkin diagrams with sign function $i \mapsto c_i$

Summary: Given root system Φ, set of simple roots $\{\alpha_i \mid i \in I\}$, and $i \mapsto c_i$, there is a unique "special Chevalley basis" $\{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ of $\mathfrak{g}$.

Can even write computer programs $\rightsquigarrow$ ChevLie at `github.com/geckmf`

$\rightsquigarrow$ "Canonical" matrix models of $\mathfrak{g}$ and corresponding Chevalley groups.

A (curious) remark. For each $\alpha \in \Phi$ there is a unique co-root $h_\alpha \in \mathfrak{h}$.

- $h_{-\alpha} = -h_\alpha$ and $h_{\alpha_i} = h_i$ for $i \in I$.
- Pre-canonical theory: initial normalisation of $\{e_\alpha\}$ such that
  - $[e_\alpha, e_{-\alpha}] = +h_\alpha$ for all $\alpha \in \Phi$ (Chevalley, Humphreys, Serre, ...), or
  - $[e_\alpha, e_{-\alpha}] = -h_\alpha$ for all $\alpha \in \Phi$ (Bourbaki, Tits, ...).
- For "special" collection $\{e_\alpha\}$, we have $[e_\alpha, e_{-\alpha}] = -(-1)^{\mathrm{ht}(\alpha)} h_\alpha$ for $\alpha \in \Phi$.

  $\rightsquigarrow$ Simple test if a given collection $\{e_\alpha\}$ is the special, canonical one.

Remaining problem. Assume $\{e_\alpha\}$ is the special, "canonical" collection.

Find explicit formulae for "structure signs" $\eta_0(\alpha, \beta) \in \{\pm 1\}$ such that

$$[e_\alpha, e_\beta] = \eta_0(\alpha, \beta)(q_{\alpha,\beta} + 1)e_{\alpha+\beta} \qquad \text{whenever} \qquad \alpha, \beta, \alpha + \beta \in \Phi.$$

Formula for $\eta_0(\alpha, \beta)$ should just depend on roots α, β and function $i \mapsto c_i$.

Alexander Lang's Master's thesis

Theorem (Lang, 2023). Assume that $\mathfrak{g}$ is simply laced, type A_n, D_n, E_6, E_7 or E_8.

Let $\alpha, \beta \in \Phi$ be such that $\alpha + \beta \in \Phi$. Then

$$\eta_0(\alpha, \beta) = \prod_{i,j \in I} c_i^{a_{ij} n_i m_j} \quad \text{where} \quad \alpha = \sum_{i \in I} n_i \alpha_i \text{ and } \beta = \sum_{j \in I} m_j \alpha_j \quad (n_i, m_j \in \mathbb{Z}).$$

Here, a_{ij} are the Cartan integers: $a_{ii} = 2$, $a_{ij} = -1$ if $i \neq j$ are joined by an edge in the Dynkin diagram, $a_{ij} = 0$ otherwise.

Proof by reduction to the case where $\alpha \in \Phi^+$, and then induction on $\mathrm{ht}(\alpha)$.

See: M. G. and A. Lang, Beitr. Algebra Geom. **66** (2025), 757–774.

Further aim of this talk:

- A new proof of the above formula which, at the same time, yields a new proof of the existence of Lusztig's canonical/special basis in the simply laced case.
- Also explicit formulae for non-simply laced types.

The Frenkel–Kac construction

Assume $\mathfrak{g}$ simply laced, type A_n, D_n, E_6, E_7 or E_8.

There is a scalar product $\langle\ ,\ \rangle$ on $\langle\Phi\rangle_{\mathbb{R}} \subseteq \mathfrak{h}^*$ such that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in \Phi$.

Let $Q := \mathbb{Z}\Phi$ and consider a function $\varepsilon\colon Q \times Q \to \{\pm 1\}$, which is bi-multiplicative:

$$\varepsilon(\alpha+\beta,\gamma) = \varepsilon(\alpha,\gamma)\varepsilon(\beta,\gamma) \qquad \text{and} \qquad \varepsilon(\alpha,\beta+\gamma) = \varepsilon(\alpha,\beta)\varepsilon(\alpha,\gamma)$$

and satisfies the “anti-symmetry condition” $\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha) = (-1)^{\langle\alpha,\beta\rangle}$ for $\alpha,\beta \in Q$.

Theorem. There is a Chevalley basis $\{h_i \mid i \in I\} \cup \{e_\alpha \mid \alpha \in \Phi\}$ such that

- $[e_\alpha, e_\beta] = \varepsilon(\alpha,\beta)e_{\alpha+\beta}$ whenever $\alpha,\beta,\alpha+\beta \in \Phi$ (here: $q_{\alpha,\beta} = 0$);
- $[e_\alpha, e_{-\alpha}] = \varepsilon(\alpha,-\alpha)h_\alpha$ for all $\alpha \in \Phi$.

(See also the book of Frenkel–Lepowsky–Meurman, 1988).

The (!) standard example for $\varepsilon\colon Q \times Q \to \{\pm 1\}$

Choose any orientation of the Dynkin diagram. For example:

E_6:

Define

$$\varepsilon(\alpha_i, \alpha_j) := \begin{cases} -1 & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ are joined by an edge and arrow points towards } j, \\ 1 & \text{otherwise.} \end{cases}$$

For arbitrary $\alpha, \beta \in Q$, write $\alpha = \sum_{i \in I} n_i \alpha_i$ and $\beta = \sum_{j \in I} m_j \alpha_j$ where $n_i, m_j \in \mathbb{Z}$; then set $\varepsilon(\alpha, \beta) := \prod_{i,j \in I} \varepsilon(\alpha_i, \alpha_j)^{n_i m_j}$. $\rightsquigarrow$ bi-multiplicative $\varepsilon\colon Q \times Q \to \{\pm 1\}$.

One also checks: $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ and $\varepsilon(\alpha, \alpha) = (-1)^{\frac{1}{2}\langle \alpha, \alpha \rangle}$ for all $\alpha \in Q$.

This implies that "anti-symmetric condition" holds, so Theorem applies.

Since $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in \Phi \subseteq Q$, we have $[e_\alpha, e_{-\alpha}] = \varepsilon(\alpha, -\alpha) h_\alpha = -h_\alpha$.
So we have Bourbaki–Tits pre-normalisation, but not a special Chevalley basis !

A slight adaptation

- The function $i \mapsto c_i$ considered earlier defines an orientation of the Dynkin diagram where each vertex is either a sink or a source. In the above example:

E_6:

So "$c_i = +1$" means sink, and "$c_i = -1$" means source.

- For such an orientation, can rewrite previous definition of $\varepsilon(\alpha_i, \alpha_j)$:

$$\varepsilon(\alpha_i, \alpha_j) = \begin{cases} -1 & \text{if } i = j \\ c_i & \text{if } i \neq j \text{ are joined by an edge} \\ 1 & \text{otherwise} \end{cases} = \begin{cases} -1 & \text{if } i = j, \\ c_i^{a_{ij}} & \text{if } i \neq j, \end{cases}$$

where a_{ij} are the Cartan integers.

- Now define: $$\varepsilon_0(\alpha_i, \alpha_j) := c_i^{a_{ij}} = \begin{cases} 1 & \text{if } i = j, \\ c_i^{a_{ij}} & \text{if } i \neq j. \end{cases} \qquad \leftarrow \text{only change !}$$

Then $\varepsilon_0(\alpha, \beta) = \prod_{i,j \in I} c_i^{a_{ij} n_i m_j}$ for all $\alpha, \beta \in Q$ $\rightsquigarrow$ Lang's formula !

Lemma. Let $\mathfrak{g}$ be simply laced and $\varepsilon_0(\alpha,\beta) = \prod_{i,j\in I} c_i^{a_{ij}n_im_j}$ for $\alpha,\beta\in Q$.

This function is bi-multiplicative and satisfies the anti-symmetry condition; so the Frenkel–Kac Theorem applies. The corresponding Chevalley basis of $\mathfrak{g}$ is special.

Proof. This follows from quite simple computations, even simpler than for the (!) "standard" example $\varepsilon\colon Q\times Q\to\{\pm 1\}$. First note: $a_{ij} = \langle\alpha_i,\alpha_j\rangle$ for $i,j\in I$, since $\mathfrak{g}$ simply laced. So, for arbitrary $\alpha,\beta\in Q$, we have $\langle\alpha,\beta\rangle = \sum_{i,j\in I} a_{ij}n_im_j$.

Now anti-symmetry:

$$\varepsilon_0(\beta,\alpha) = \prod_{i,j\in I} c_i^{a_{ij}m_in_j} = \prod_{i,j\in I} c_j^{a_{ij}n_im_j} = \prod_{i,j\in I} (-c_i)^{a_{ij}n_im_j} = \varepsilon_0(\alpha,\beta)(-1)^{\sum_{i,j\in I} a_{ij}n_im_j},$$

and exponent of (-1) equals $\langle\alpha,\beta\rangle$ as required.

Finally, $i\in I$ and $\beta\in\Phi$ such that $\alpha_i+\beta\in\Phi$. Since Φ simply laced, $\langle\alpha_i,\beta\rangle = -1$.

Then $\varepsilon_0(\alpha_i,\beta) = \prod_{j\in I} c_i^{a_{ij}m_j} = c_i^{\sum_{j\in I} a_{ij}m_j} = c_i^{\langle\alpha_i,\beta\rangle} = c_i$. So $[e_{\alpha_i}, e_\beta] = c_ie_{\alpha_i+\beta}$. $\square$

The general case: non-simply laced types

Let $A = (a_{ij})_{i,j\in I}$ be the Cartan matrix for Φ in general; not symmetric any more !

But symmetrisable: Let $d_i \in \mathbb{Z}_{\geqslant 1}$ be minimal such that $d_i a_{ij} = a_{ji} d_j$ for all $i, j \in I$.

Could try the following generalisation of Lang's formula.

$$\varepsilon_0(\alpha,\beta) := \prod_{i,j\in I} c_i^{d_i a_{ij} n_i m_j} = \pm 1 \qquad \text{for} \qquad \alpha, \beta \in Q = \mathbb{Z}\Phi.$$

Simple examples show that this is not good enough to describe the "structure signs" $\eta_0(\alpha,\beta)$ for $\mathfrak{g}$! Instead, de-construct into summation and exponent part:

Let $\vec{\mathcal{I}}(A)$ be the set of all pairs $(i,j) \in I \times I$ such that $i \neq j$, $a_{ij} \neq 0$ and $c_i = -1$.

Then $\varepsilon_0(\alpha,\beta) = (-1)^{\vec{\rho}(\alpha,\beta)}$ where $\vec{\rho}(\alpha,\beta) := \sum_{(i,j)\in\vec{\mathcal{I}}(A)} d_i a_{ij} n_i m_j \in \mathbb{Z}$.

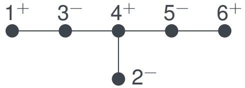
RINGEL (1990) and RYLANDS (2000): For the "structure signs" of $\mathfrak{g}$ in general, it seems both $\vec{\rho}(\alpha,\beta)$ and $\vec{\rho}(\beta,\alpha)$ are needed, and also not just their values mod 2.

Types B_n, C_n, F_4 and G_2

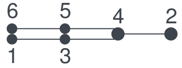
Lie algebras of these types can be obtained from simply laced ones by “folding”.

- Let $\tilde{\mathfrak{g}}$ be a simple Lie algebra with diagram of type $\boxed{A_{2n-1},\ D_{n+1},\ E_6 \text{ or } D_4}$.
- Let $\tilde{I} \leftrightarrow$ Dynkin diagram of $\tilde{\mathfrak{g}}$ and $\tilde{A} = (\tilde{a}_{ij})_{i,j\in\tilde{I}}$ symmetric Cartan matrix.
  There are diagram symmetries given by a permutation $i \mapsto i'$ of $\tilde{I}$ such that
  $$\tilde{a}_{ij} = \tilde{a}_{i'j'} \ \text{(all } i,j) \qquad \text{and} \qquad \tilde{a}_{ii'} = 0 \ \text{(if } i \neq i').$$

Example E_6:



$1' = 6,\ 3' = 5$
$2' = 2,\ 4' = 4$

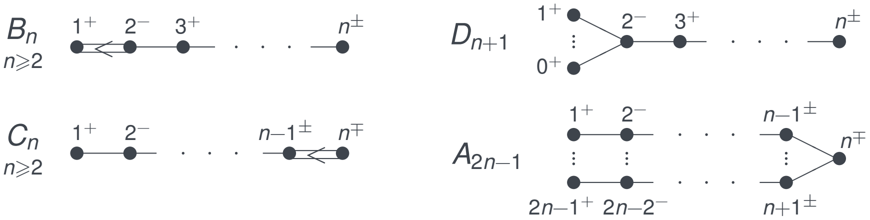


orbits: $\{1,6\}, \{3,5\}, \{4\}, \{2\}$

- The permutation $i \mapsto i'$ extends to an automorphism $\tau\colon \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ such that
  $$\tau(\tilde{e}_{\pm\alpha_i}) = \tilde{e}_{\pm\alpha_{i'}} \qquad \text{and} \qquad \tau(\tilde{h}_i) = \tilde{h}_{i'} \qquad \text{for all } i \in \tilde{I}.$$
- $\mathfrak{g} := \{x \in \tilde{\mathfrak{g}} \mid \tau(x) = x\}$ is a simple Lie algebra of type $\boxed{C_n,\ B_n,\ F_4,\ G_2 \text{ (resp.)}}$.

Use this to reduce computation of “structure signs” for $\mathfrak{g}$ to known case of $\tilde{\mathfrak{g}}$.

Folding for type B_n and C_n



Proposition (G., `arxiv:2602.19632`). Assume $\mathfrak{g}$ of type B_n or C_n, as above.

The "structure signs" $\eta_0(\alpha, \beta)$ for the special Chevalley basis of $\mathfrak{g}$ are given as follows, where $\alpha, \beta \in \Phi^+$ are such that $\alpha + \beta \in \Phi$.

- If $\mathfrak{g}$ is of type B_n, then $\vec{\rho}(\alpha, \beta) \in 2\mathbb{Z}$ and $\eta_0(\alpha, \beta) = (-1)^{\vec{\rho}(\alpha,\beta)/2}$.
- If $\mathfrak{g}$ is of type C_n, then $\vec{\rho}(\alpha, \beta) - \vec{\rho}(\beta, \alpha) \in \{\pm 1, \pm 2, \pm 3\}$ and

$$\eta_0(\alpha, \beta) = \begin{cases} 1 & \text{if } \vec{\rho}(\alpha, \beta) > \vec{\rho}(\beta, \alpha), \\ -1 & \text{if } \vec{\rho}(\alpha, \beta) < \vec{\rho}(\beta, \alpha). \end{cases}$$